Bohr-Riemann surfaces

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1. Generalized plane Δ

Let Γ be a subgroup of the group of real numbers $\mathbb{R}$ and let G be the group of characters of Γ: $G = \hat{\Gamma}$. By Pontryagin duality Theorem $\hat{G} \cong \Gamma$.
Using G we construct the space Δ:

$$\Delta = G \times [0, \infty)/G \times \{0\}.$$

(construction is due to Arens and Singer [1]).
Two cases:

- $\Gamma \cong \mathbb{Z}$,
- Γ is dense in $\mathbb{R}$ in a Euclidean topology τ.

In case $\Gamma \cong \mathbb{Z}$ we get $G = \mathbb{T}$ and $\Delta \cong \mathbb{C}$, where $\mathbb{T}$ is the unit circle of the complex plane $\mathbb{C}$.
Theory of generalized analytic functions (defined later) in this case is similar to its classical prototype.

1. Generalized plane Δ

So we suppose that Γ is dense in $\mathbb{R}$ and consider the space (generalized plane)

$$\Delta = G \times [0, \infty)/G \times \{0\}.$$

The space Δ consists of the elements $(\alpha, r), \alpha \in G, r > 0$ and the element $* = G \times \{0\}$ – null element of Δ.
Denote $\Delta^0 = \Delta \backslash \{*\}$ – punctured generalized plane.

The space Δ can be canonically identified with the big plane $\mathcal{C} = \{\alpha r : \alpha \in G, r \in [0, \infty)\}$ – an analogue of the complex plane $\mathbb{R}$ composed of the homomorphisms

$$\alpha r : \Gamma \to \mathbb{C} : a \mapsto \alpha(a) r^a, a \in \Gamma.$$

Thus, we can assume that $\Delta = \mathcal{C}$ and for a "polar decomposition" $s = \alpha r$ of the element $s \in \Delta$ we say that the number $r = |s|$ is a modulus of s.

2. Generalized analytic functions

Let $\chi^a \in \hat{G}$ be a character corresponding to the element $a \in \Gamma$. Let us define $\Gamma_+ = \{a \in \Gamma : a \geq 0\}$. Then each character $\chi^a, a \in \Gamma_+$ can be extended to a continuous function φ^a on Δ setting for $s = \alpha r$

$$\varphi^a(s) = \chi^a(\alpha) r^a$$

with $\chi^a(\alpha) = \alpha(a)$.

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Definition 2.1

Let D be an open set in Δ. Continuous function f on Δ is called generalized analytic function if each $s \in D$ has a neighbourhood $U \subset D$, $s \in U$, such that f on U can be uniformly approximated by linear combinations of the functions $\varphi^a, a \in \Gamma_+$.

The set of all generalized analytic functions on D is denoted $\mathcal{O}(D)$.

2. Generalized analytic functions

Using the fact that the space Δ^0 locally has a structure of the form $V \times W, V \subset G_a, W \subset \mathbb{C}$, with $G_a = \{\alpha \in G; \alpha(a) = 1\}$, $a \in \Gamma$, ([2], pp 10-11), we give the following definition.

Definition 2.2

Let D be an open set in Δ. A subset $K \subset D$ is called a thin set if it is closed in D and the following conditions hold:

1. for each $s \in D, s \neq *,$, there exist a neighbourhood $U \subset D, U = V \times W$ of s and a non-zero function $f \in \mathcal{O}(U)$, such that f is zero on $K \cap U$, and for each $\alpha \in V$ f is not identically zero on $W_\alpha = \{\alpha\} \times W$.
2. if $* \in D$ then there exists a non-zero function $f \in \mathcal{O}(\Delta_r), \Delta_r \subset D$, which is zero on $\Delta_r \cap K$, where $\Delta_r = \{s \in \Delta; |s| \leq r\}$ – is a generalized disc of radius r.

3. Bohr-Riemann suraces

Definition 3.1

The mapping $\pi : Y \to X$ between topological spaces Y and X is called (in general, branched) covering if it is continuous, open and discrete, i.e. for each $x \in X$ the set $\pi^{-1}(x)$ is a discrete set in Y.

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Definition 3.2

The mapping $\pi : Y \to X$ between topological spaces Y and X is called unbranched covering if each point $x \in X$ has a neighbourhood $U \ni x$ such that

$$\pi^{-1}(U) = \bigcup_{i \in \mathcal{A}} U_i$$

– disjoint union of open sets in Y, and all restrictions $\pi|_{U_i} : U_i \to U$ are homeomorphisms.
If the set $\mathcal{A}$ is finite then π is called unbranched, finite-sheeted covering

Definition 3.3

Topological space X is called a Bohr-Riemann surface over Δ if there exist a thin set $K \subset \Delta$ and a covering $\pi : X \to \Delta$ such that the restriction of π to the set $X^* = X \backslash \pi^{-1}(K)$ is an unbranched, finite-sheeted covering of the set $\Delta^* = \Delta \backslash K$.

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Remark: We usually suppose that $* \in K$ and consider the coverings over $\Delta^0 = \Delta \backslash \{*\}$.
Note that as the set $K = \{*\}$ is obviously a thin set, we get that if there exists a covering $\pi : X \to \Delta$ such that $\pi : X^* \to \Delta^0 = \Delta^0 = \Delta \backslash \{*\}$, with $X^* = X \backslash \pi^{-1}(*)$, is an unbranched, finite-sheeted covering of Δ^0 then X is a Bohr-Riemann surface over Δ.

We can similarly define a Bohr-Riemann surface over the open subsets of Δ.

Definition 3.4

Suppose that D is an open subset of Δ. Topological space X is called a Bohr-Riemann surface over D if there exist a thin set $K \subset D$ and a covering $\pi : X \to D$ such that the restriction of π to the set $X^* = X \backslash \pi^{-1}(K)$ is an unbranched, finite-sheeted covering of the set $D^* = D \backslash K$.

4. Group structures on the Bohr-Riemann surfaces

Theorem 4.1, (Grigoryan, Gumerov, Kazantsev, [3])

Let $p : X \to G$ be an n-fold covering of a compact solenoidal group G by a connected topological space X. Then there exists a group structure in X turning $p : X \to G$ into a homomorphism between compact abelian groups.

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Theorem 4.2, [4]

Let $\pi : X^0 \to \Delta^0$ be an n-fold, unbranched covering of the punctured generalized plane Δ^0 by a connected topological space X^0.Then there can be defined a group structure on X^0 turning π into a group homomorphsim between X^0 and Δ^0. The group X^0 is then topologically isomorphic to the Cartesian product $G_1 \times (0, +\infty)$ with G_1 being a compact subgroup of X^0.

5. Analytic curves and equivalent points on the Bohr-Riemann surfaces

Let us consider the mapping $\alpha : \mathbb{R} \to G : t \to \alpha_t$, where $\alpha_t(a) = e^{iat}, a \in \Gamma$. Using density of Γ in $\mathbb{R}$ it can be shown that α is injective and $\alpha(\mathbb{R})$ is dense in $\mathbb{R}$. Having α we now define a map

$$\varphi : \mathbb{C} \to \Delta^0 : z = t + iy \mapsto \varphi_z = \alpha_t e^{-y}.$$

Note that $\varphi(\mathbb{C})$ is dense both in Δ^0 and in Δ.
The set of the form $\mathbb{C}_s := s\varphi(\mathbb{C})$ is called a plane in Δ^0 passing through the point $s \in \Delta^0$; $\mathbb{C}_0 := \mathbb{C}_{\varphi(0)}(= \varphi(\mathbb{C}))$.

5. Analytic curves and equivalent points on the Bohr-Riemann surfaces

Let us now consider a path in Δ^0, i.e. continuous map $\gamma : I = [0, 1] \to \Delta^0$.

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A path $\gamma(I) \subset \Delta^0$ is called analytic if it is entirely contained in some plane $\mathbb{C}_s, s \in \Delta^0$.

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Path lifting property

Given a covering map $\pi : X^0 \to \Delta^0$, a path $\gamma : I \to \Delta^0$ with $\gamma(0) = \sigma$, and a point $x \in \pi^{-1}(\sigma)$, there is a unique path $\hat{\gamma}(I) \subset X^0$ starting at the point x and lifting γ, i.e. $\hat{\gamma}(0) = \sigma$ and $\gamma(t) = \pi \circ \hat{\gamma}(t), t \in I$.

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Definition 5.2

A path $\hat{\gamma}(I) \subset X^*$ is called analytic if it is a lifting of an analytic path from Δ^*.

5. Analytic curves and equivalent points on the Bohr-Riemann surfaces

Let us define the notion of equivalent points on the sets $\pi^{-1}(s), s \in \Delta^*$.

Definition 5.3

Suppose that X is a Bohr-Riemann surface over Δ and $\pi : X \to \Delta$ is a covering map. Two points $w_1, w_2 \in \pi^{-1}(s)$ are called equivalent $(w_1 \sim w_2)$ if there exists an analytic path $\hat{\gamma}(I) \subset X^*$ such that $w_1 = \hat{\gamma}(0)$ and $w_2 = \hat{\gamma}(1)$.

It can be shown that this is a well defined equivalence relation.

5. Analytic curves and equivalent points on Bohr-Riemann surfaces

On a Bohr-Riemann surface X^* consider the function $\nu : X^* \to \mathbb{Z}_+$ with

$$\nu(w_0) = card\{w \in \pi^{-1}(\pi(w_0)) : w \sim w_0\}, w_0 \in X^*$$

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Theorem 5.1, [5]

The function $\nu : X^* \to \mathbb{Z}_+$ is locally constant on X^*.

5. Analytic curves and equivalent points on the Bohr-Riemann surfaces

We now pass to the algebraic version of the theory.
Let

$$p(s,x) = x^n + f_1(s)x^{n-1} + \dots + f_n(s)$$

be a polynomial with generalized analytic coefficients: $f_i \in \mathcal{O}(\Delta^0), i = \overline{1,n}$ and with discriminant $d_p(s)$. Then $d_p(s)$ is also generalized analytic function $d_p \in \mathcal{O}(\Delta^0)$. Denote $N_p = N(d_p)$ – set of zeros of the function d_p. Then either N_p is a discrete set in Δ^0 or $N_p = \Delta^0$. We suppose that the first case holds and in this case N_p plays the role of a thin set.

5. Analytic curves and equivalent points on the Bohr-Riemann surfaces

Let us consider the space

$$\Delta_p^0 = \{(s,x) \in \Delta^0 \times \mathbb{C} : p(s,x) = 0\},$$

and the covering

$$\pi : \Delta_p^0 \to \Delta^0 : (s,x) \mapsto s.$$

The restriction $\pi|_{\Delta_p^*} : \Delta_p^* = \pi^{-1}(\Delta^*) \to \Delta^*$ is then an unbranched n-fold covering over $\Delta^* = \Delta^0 \setminus N_p$. Thus, Δ_p^0 is a Bohr-Riemann surface.

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Again we define the notion of equivalent points on Δ_p^* and the function $\nu : \Delta_p^* \to \mathbb{Z}_+$ which counts for a given $w \in \Delta_p^*$ the number of its equivalent points.

Theorem 5.2, [5]

The function $\nu : \Delta_p^* \to \mathbb{Z}_+$ is locally constant on Δ_p^*.

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